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# Zeros of the Fourier Transform of a Distribution

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## 1. INTRODUCTION

Let $\mathscr{S}$ denote the Schwartz space of functions $\phi: \mathbb{R} \to \mathbb{C}$ and let $\mathscr{S}'$ be the space of tempered distributions [4, p.19]. By the Paley–Wiener theorem the Fourier transform of a tempered distribution with support $[-\sigma, \sigma]$ is an entire analytic function satisfying a growth condition [4, p. 21].

The trigonometric polynomial $f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$, $c_n \in \mathbb{C}$, is the Fourier transform of a tempered distribution with support $[-N, N]$. In section 2 we prove:

THEOREM 1. *The trigonometric polynomial* $f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$, $c_n \in \mathbb{C}$, *is nonzero on at least one open interval of the real axis of length* $d > \pi/N$, *except for the exceptional case where* $f(\theta) = c \sin N(\theta - \alpha)$, $0 \leqslant \alpha < 2\pi$, $c \in \mathbb{C}$.

*The exceptional case arises as the Fourier transform of the distribution,* $T(u) = e^{-iN\alpha}\delta(u - N) - e^{iN\alpha}\delta(u + N)$, *where* $\delta$ *is the Dirac distribution.*

A distribution with support $\{0\}$ is a finite linear combination of derivatives of $\delta$ [5, p. 387]. Under the Fourier transform differentiation corresponds to multiplication. Hence if $p$ is a polynomial, then $f(x) = p(x) \sin \sigma(x - \alpha)$ is the Fourier transform of a distribution with support $\{\sigma\} \cup \{-\sigma\}$. The distance between consecutive zeros of $f$ is less than or equal to $\pi/\sigma$.

We ask if apart from these exceptional cases, every entire function which is the Fourier transform of a distribution with support $[-\sigma, \sigma]$ is nonzero on at least one open interval of the real axis of length $d > \pi/\sigma$. Theorem 2 is the statement of this result for the special case that $f$ is the Fourier transform of a function $\phi \in L^2[-\sigma, \sigma]$. An immediate corollary of Theorem 2 is the well-known result that if $f(n\pi/\sigma) = 0$, $n \in \mathbb{Z}$, then $f \equiv 0$.



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This corollary can be generalised. A sampling theorem of Campbell [2], implies that if $f$ is the Fourier transformation of a distribution with support $(-\sigma, \sigma)$ and $f(n\pi/\sigma)=0$, $n \in \mathbb{Z}$, then $f \equiv 0$.

## 2. Special Cases

We now prove some special cases of the conjecture of Section 1.

*Proof of Theorem* 1. Under the transformation $z=e^{i\theta}$, $f(\theta)=p(e^{i\theta})$, where

$$\begin{aligned} p(z) &= \sum_{n=-N}^{N} c_n z^n \\ &= z^{-N}(c_{-N} + c_{-N+1} z + \cdots + c_N z^{2N}). \end{aligned}$$

Zeros of $f$ on the real axis correspond to roots of $p$ on the unit circle and distances between zeros on the real axis correspond to angular displacements between roots on the unit circle.

Since $p$ has $2N$ roots which can lie anywhere in the complex plane, the result is clear. The exceptional case arises when the $2N$ roots are equally spaced on the unit circle.

For the next theorem we require the following inequality.

*Wirtinger's Inequality.* Suppose $f$ is a complex-valued function defined on the interval $[a, b]$ with $f \in C^1\ [a, b]$ and $f(a)=f(b)=0$. Then

$$\int_a^b |f(x)|^2\, dx \leqslant \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2\, dx,$$

with equality if and only if

$$f(x) = c \sin\left(\frac{(x-a)}{(b-a)}\right), \qquad c \in \mathbb{C}.$$

*Proof.* The theorem is proved for $f$ real-valued in [3, p. 185]. It follows for $f$ complex-valued by considering the real and imaginary parts separately.

**Theorem** 2. *Suppose $f \not\equiv 0$ is the Fourier transform of $\phi \in L^2[-\sigma, \sigma]$. Then $f$ is nonzero on at least one open interval of the real axis of length $d > \pi/\sigma$.*

*Proof.* By the classical Paley–Wiener theorem $f \in L^2(\mathbb{R})$. We need only consider the case in which $f$ has infinitely many zeros on the real axis.

Suppose the distance between any consecutive pair of zeros is less than or equal to $M$. The theorem follows if we show that $M > \pi/\sigma$.

Suppose $a$ and $b$ are consecutive zeros. Then by Wirtinger's inequality

$$\int_a^b |f(x)|^2 \, dx < \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \, dx.$$

The inequality is strict because, by the identity theorem, if $f$ were the sine function on $(a, b)$ it would also be the sine function on $\mathbb{R}$. Summing over all consecutive pairs of zeros, we have

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \frac{M^2}{\pi^2} \int_{-\infty}^{\infty} |f'(x)|^2 \, dx,$$

and hence $\|f\| < (M/\pi) \, \|f'\|$. On the other hand, by the $L^2$ version of Bernstein's theorem [1, p. 211], $\|f'\| \leqslant \sigma \, \|f\|$, and we may conclude that $M > \pi/\sigma$.

*Remarks.* (i) A weaker version of Theorem 2 appears in [6] and its connection with Levinson's theorem on the density of zeros is discussed there.

(ii) The example $f(x) = (\sin \sigma x)/(\sigma x)$ shows that $f$ may be nonzero on precisely one interval of length $d > \pi/\sigma$. Also it is straightforward to construct from $f$ a new function $g$, which satisfies the hypotheses of Theorem 2, and has the $2N$ zeros, $x = \pm n\pi/\sigma$, $1 \leqslant n \leqslant N$, relocated at an equal spacing $d = 2(N+1)\,\pi/(2N+1)\,\sigma$ on the open interval $(-(N+1)\,\pi/\sigma, (N+1)\,\pi/\sigma)$. Hence $d$ can be made arbitrarily close to $\pi/\sigma$ as $N \to \infty$.

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